Distributed Systems Approach to the Identification of Flexible Structures

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This paper presents a distributed parameter estimation scheme and investigates its computational merit for three idealized examples of the identification of large flexible structures. The method retains the distributed nature of the structure throughout the development of the algorithm and a finite-element approximation is used only to implement the algorithm. This approach eliminates many problems associated with the model truncation used in other methods of identification. The identification problem is formulated in Hilbert spaces and an optimal control technique is used to minimize a weighted least squares of error between the actual and the model data. A variational approach is used to solve the problem. A costate equation, gradients of parameter variations and conditions for optimal estimates are obtained. Computer simulation studies are conducted using flexible beam models as examples. Numerical results show a close match between the estimated and true values of the parameters.

Introduction

WITH the advent of a space transportation system and NASA's present emphasis on the deployment of a permanent space station, the area of large space structures (LSS) has received much study and attention in these last few years. By necessity, these large structures are flexible and very complex in nature. Their dynamic characteristics in space cannot be determined through ground testing alone and must be derived from both ground testing and testing in space. There is also a basic need for a control system to obtain an accurate set of system parameters so as to tune itself to give proper and stable system response.[1] Often, the LSS is divided into substructures, such as equivalent flexible beams, plates, membranes, etc., and tested individually. Motivated by the identification of large flexible structures, we present a distributed parameter estimation scheme and investigate its computational merit for the three idealized examples. The basis for the development is that a large space structure is a distributed system and its parameters are either constants or functions of spatial variables. Moreover, the joint-dominated structures can be approximated as equivalent flexible beams or plates with distributed parameters for control purposes. These parameters are to be determined from the knowledge of either experimental or real data in space.

Recently, there have been several conferences, workshops, reports, and publications by NASA, AIAA, AAS, and IEEE devoted to LSS's problems. From these conferences and journal publications, two distinct approaches to the solution of LSS's parameter identification problems have emerged. They are

1) A finite-dimensional design approach where a structural model is first truncated to a reduced-order model and an estimator is designed based on the reduced-order model.

2) An approach where an infinite-dimensional (distributed) model is retained as long as possible throughout the development of an estimation algorithm and the necessary truncation is carried out only to implement the algorithm.

Table 1 is a list of some of the work done along these two approaches to the parameter identification problems for LSS's.[2–16] There are also two excellent survey papers for the distributed system parameter identification problem (DSPIP) for the general cases written by Polis and Goodson[17] and Polis.[18] It is obvious from these surveys that by far the most effort put into the study of the identification problem to date has been based on the finite-dimensional (finite-element or finite-difference) models, which can be addressed using ordinary differential equation (ODE) parameter estimation theory. The main disadvantage of this approach is the high dimensionality involved, which greatly restricts the use of identification algorithm developed for ODE. Therefore, in this paper the second approach is used, i.e., the distributed nature of the structure [continuum or partial differential equation (PDE) model] is retained as long as possible before the parameter identification algorithm is implemented numerically. The major motivation behind the use of a continuum model is that, if one does not use a continuum approach, there might be a problem which has been overlooked and a design may not work when implemented on a real system.[19] The PDE's are free from an arbitrary dimensionality imposed through discretization and reduce at least one source of error that is introduced externally into the model. In the continuum model approach, the discretization or truncation is carried out in a later stage to implement the algorithm. It needs to be pointed out that the truncation is carried out at a different point in the problem, whereas truncating in the finite-dimensional approach, in which one throws away modes, implies throwing away some of the information.

In this paper, the parameter identification problem for a flexible beam representing a generic LSS component is formulated in an abstract setting in Hilbert spaces and an optimal control technique is then used to solve the identification problem. The basic approach is the use of an iterative scheme to find the parameter values (mass, flexural rigidity, damping coefficient, etc.) that minimize the error between the simulated model output and the measured data. A variational approach

Table 1 Literature surveyed on the parameter estimation of large space structures

Reference	Approach
2–13	A finite-dimensional design approach where the structural model is truncated and the estimator is designed based on the reduced-order model
13–16	An infinite-dimensional design approach where the PDE model is retained as long as possible and truncation is carried out only after the estimation algorithm is developed

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is used to solve the problem and the necessary conditions are obtained. A steepest descent technique method is used for the convergence of error criterion. In the implementation stage of the algorithm, infinite-dimensional Hilbert spaces are approximated by finite-dimensional spaces using the finite-element method.

In sections that follow, a mathematical framework for the parameter identification of a flexible beam is developed. Then, an abstract formulation of the identification problem follows. An infinite-dimensional algorithm is then developed for the solution of the identification problem. At the end of this paper, numerical results of the parameter identification of vibrating beams in three specific cases are presented to demonstrate the method developed.

Parameter Identification Problem

In this section, a mathematical framework for the parameter identification problem for a flexible beam representing a generic component of the large space structures (LSS) is presented. The basis for the development of this format is that a large space structure is a distributed system that is controlled by a few localized actuators and that the algorithm is to be implemented in an on-board computer of limited size. The LSS system can be partitioned into a number of components, each of which can be modeled by a simple partial differential equation. The motivation behind this partitioning of a LSS into localized sections is that the partial differential equation for a substructure describes the behavior of the spacecraft within a local neighborhood of each point. Since sensors measure and disturbances affect local variables, a reasonable job of parameter identification may be accomplished with "local" parameter identification.

A Distributed System Model

The large space structures can be approximately decomposed into equivalent flexible bodies of simple structures such as strings, beams, membranes, plates, and discrete point masses and rigid bodies. The dynamic behavior of the equivalent structures can be described by a system of partial differential equations of the form

$$m(x)\frac{\partial^2 u}{\partial t^2}(x,t) + D_0\frac{\partial u}{\partial t}(x,t) + A_0 u(x,t) = F(x,t)$$

$$x \in \Omega, \quad t \in (0,T) \tag{1}$$

where $u(x,t)$ represents instantaneous displacement of the structure on a spatial domain Ω off its equilibrium position due to transient disturbances or the applied force function $F(x, t)$. The displacement can be translational or rotational and the force can be generalized to include torque as well. The mass per unit length $m(x)$ is positive and bounded on Ω.[1]

The structural stiffness is determined by a time-invariant, symmetric, nonnegative differential operator A_0. The domain $D(A_0)$ of A_0 contains all of the smooth functions satisfying the boundary conditions and is dense in the infinite-dimensional Hilbert space $H_0 = L_2(\Omega)$, the space of square integrable functions, with the usual inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$.

The damping occurs due to material properties and construction techniques. It is represented by the differential operator D_0. Its nature in space is poorly known and it is believed to be very light. Thus, in many LSS analyses, damping is neglected.

The applied force distribution is

$$F(x,t) = F_B(x,t) + F_C(x,t) + F_D(x,t) \tag{2}$$

where F_D represents the external disturbance forces on the system (and possible nonlinearities) and F_C the control forces due to M actuators, as

$$F_C(x,t) = B_0 f = \sum_{i=1}^{M} b_i(x) f_i(t) \tag{3}$$

where the actuator amplitudes are $f_i(t)$ and the actuator influence functions are $b_i(x)$ in H_0. These are usually localized or point devices so that $b_i(x)$ is often approximated by $\delta(x - x_i)$ where x_i is a point in Ω. However, they do not need to be point sources. Finally, F_B represents the boundary forcing function. It normally represents the forces that couple the local subsystems modeled by Eq. (1) to other subsystems modeled by an analogous equation. In general, it can be represented as

$$F_B(x,t) = B_b g = \sum_{i=1}^{N} b_i(x) g_i(t) \tag{4}$$

where $g_i(t)$ is a boundary force amplitude that can be either a force or a moment due to another subsystem coupled at the boundary. Consequently, $b_i(x)$ can be approximated either by $\delta(x - x_i)$ or $(\partial/\partial x)\,\delta(x - x_i)$, respectively, where x_i is a point in the boundary $\partial\Omega$.[25] We show $g_i(t)$ to be a function of time only for rotational compaction and it is clear that F_B will depend upon the deformation states (and derivatives thereof) of the locally coupled subsystems.

Since the subsystem of Eq. (1) is coupled to other subsystems through the boundary constraint forces, it is necessary to solve the family of PDE's as well as the rigid-body equations (ODE's) simultaneously. The vibration of any substructure obviously excites its neighbors (and vice versa) through boundary or joint forces and moments. These joint forces are not necessarily actuators and, of course, are not fully under local control since they are associated with the type of boundary conditions (pinned, clamped, etc.) existing between adjacent substructures and, obviously, they depend upon the motion of adjacent substructures.[25] This paper, however, considers the single PDE case only as a first step down a path toward a method useful in applications.

The boundary conditions depend on the structural configurations. The initial conditions are given by initial displacement or initial velocity as

$$u(x,0) = u_0$$

$$u_t(x,0) = \frac{\partial u}{\partial t}(x,0) = u_1, \qquad x \in \Omega \tag{5}$$

If the system is observed by P sensors, the output vector is in the form

$$y = C_0 u + E_0 u_t \tag{6}$$

where $y_j(t) = \langle c_j, u\rangle + \langle e_j, u_t\rangle$, $1 \le j \le P$, and c_j and e_j are influence functions of position and velocity sensors, respectively, in H_0. Again, these are usually localized or point devices.

Possible disturbance forces include thermal gradients, gravity gradients, aerodynamic forces due to atmospheric effects, and meteorite collisions, as well as on-board disturbances due to pumps, motors, crew motions, etc. Control actuators can be thrusters, control-moment gyros, momentum wheels, and interelement devices, while the sensors might be mixture of position, velocity, and acceleration devices. In this paper, a disturbance-free force function is considered, i.e., $F_D = 0$.

Basic Problem Formulation

Equation (1) is a linearized system model of an equivalent flexible beam as a generic component of a large space structure and, thus, the model does not represent the true model. Furthermore, parameters in the equivalent model are not fully known to a designer. Therefore, an identification method should be developed to estimate these parameters for the best representation of the system.

Let $q(x)$ be the vector of system parameters in Eq. (1) and Q an admissible set of parameters. Then, the parameter identification problem is to determine the parameter vector $q^*(x) \in Q$, which minimizes

$$J(q) = \frac{1}{2T}\int_0^T (y - z)^T R(t)(y - z)\,\mathrm{d}t \tag{7}$$

where z is the measurement of output vector y given as

$$z(x,t) = y(x,t) + e(x,t) \tag{8}$$

with a measurement error $e(x,t)$. Also, it is defined that

$$(y - z)^T R(t)(y - z) = \int_\Omega [y(x,t) - z(x,t)]^T \times R(x,t)[y(x,t) - z(x,t)] \, \mathrm{d}x \tag{9}$$

where the matrix R is positive-definite and symmetric. In the above equations, y^T denotes the transpose of y and T alone represents the final time.

Infinite-Dimensional Formulation

To develop an estimation algorithm, it is convenient to first introduce the notations necessary to define suitable function spaces for the estimation problem. In distributed systems governed by partial differential equations as in Eq. (1), the state is a function, at each point in time, defined on a given spatial domain or, alternatively, the state is a point in an infinite-dimensional (function) space. Therefore, the distributed system of Eq. (1) can be viewed as an abstract second-order differential equation in an infinite-dimensional space. Let H be a Hilbert space with inner product $\langle \phi, \varphi \rangle = \int_\Omega \phi(x)\varphi(x) \, \mathrm{d}x$ and associated norm $\|\cdot\|$. It is also defined that

$$H = H_0(\Omega) = L_2(\Omega) \tag{10}$$

and

$$V = H_2(\Omega), \qquad V \text{ separable} \tag{11}$$

where $L_2(\Omega)$ is the space of square integrable functions on Ω and $H_m(\Omega)$ is the usual Sobolev space of degree m defined over the domain Ω. If V is a dense subspace of H and H a pivot space, then there exist the continuous embeddings $V \subset H \subset V'$, where V' is the space of continuous linear functionals on V. The norm on V is defined as $\|\cdot\|$.

Let $L_2(0,T;V)$ be defined for any function $f(t)$ such that $f(t) \in V$, $t \in [0,T]$, and

$$\left(\int_0^T \|f(t)\|^2 \, \mathrm{d}t \right)^{1/2} < \infty \tag{12}$$

In a similar manner, $L_2(0,T;V')$ may also be defined. Hence, if $f(t) \in L_2(0,T;V)$ and $A(t) \in L(V;V')$, then $t \in [0,T]$ is a linear operator from V into V' and $A(t)f(t)$ is a function such that

$$t \to A(t)f(t) \in V' \tag{13}$$

It may be verified that this function is measurable and satisfies

$$\|A(t)f(t)\|_{V'} \le c \|f(t)\|_V \tag{14}$$

for some constant $c > 0$.

The identification problem developed in the last two subsections can now be rewritten in a linear operator form as

$$\frac{\partial^2}{\partial t^2} u(t) + D(q) \frac{\partial}{\partial t} u(t) + A(q)u(t) = B(q)f(t) \qquad \text{in } (0,T]$$

$$u \in L_2(0,T;V), \qquad \frac{\partial u}{\partial t} \in L_2(0,T;H) \tag{15}$$

where $f(t)$ is given in $L_2(0, T; V)$ and the initial conditions are

$$u(0) = u_0, \qquad u_0 \text{ given in } V$$

and

$$\frac{\partial}{\partial t} u(0) = u_1, \qquad u_1 \text{ given in } H \tag{16}$$

The operators in Eq. (15) are given by

$$D(q) = \frac{1}{m(x)} D_0, \quad A(q) = \frac{1}{m(x)} A_0, \quad B(q) = \frac{1}{m(x)} B_0 \tag{17}$$

It may be pointed out here that the letter u is used in two different ways: in Eq. (1), $u = u(x,t)$ and $(x,t) \in \Omega \times [0,T]$, whereas in Eq. (15), $u(t)$ or $u(t;q)$, if dependence on q is emphasized, is an element of the function space V. The state space V is defined in such a way that any element in V will satisfy the boundary conditions defined for the structure.

The output function is

$$y(t) = Cu(t) \tag{18}$$

where $C \in L(V;Y)$ is a continuous linear operator from V into Y, Y being another Hilbert space. For a system with continuous distributed data, it is appropriate to define $Y = L_2(\Omega)$. However, $Y = R^N$ is applicable to situations where only discrete measurements are available.

The identification problem can now be formulated as an abstract problem of determining the parameter vector $q^*(x) \in Q$ that minimizes

$$J(q) = \frac{1}{2T} \int_0^T [y(t) - z(t)]^T R(t)[y(t) - z(t)] \, \mathrm{d}t \tag{19}$$

where $z(t)$ is the observed data belonging to Y and $y(t)$ is the solution of Eqs. (15–18).

Development of Infinite-Dimensional Identification Algorithm

The identification problem defined by Eqs. (15–18) can be viewed as an optimal control problem, where the parameter vector $q(x)$ is considered as control. Since the parameters are in the coefficients of Eq. (15), the problem can be approached as an optimal coefficient control problem.[20] The general nonlinear control problem has been solved using a variational approach by Lions[21] and applied to a class of identification problems by Chavent[22,23] and Lee[13] and Lee and Hossain.[24] The approach to be presented in this paper will be the generalization of these developments to a broader class of second-order functional differential equations. The main result is summarized below.

Theorem: Given a state equation (15) with initial conditions given by Eq. (16) and the cost function by Eq. (19) with $y(t)$ satisfying Eq. (18), then the optimal parameter vector q^* satisfies the state equations (15) and (16) and the following system of equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} p(t) - D^* \frac{\mathrm{d}}{\mathrm{d}t} p(t) + A^* p(t) = -\frac{1}{T} C^T R(Cu - z) \tag{20}$$

with the final conditions

$$p(T) = \frac{\mathrm{d}}{\mathrm{d}t} p(T) = 0 \tag{21}$$

and, moreover, the first variation of an augmented cost functional is

$$\delta J_a = \int_0^T p^T \frac{\partial}{\partial q} \left[D \frac{\mathrm{d}u}{\mathrm{d}t} + Au - Bf \right] \delta q \, \mathrm{d}t = 0 \tag{22}$$

where $p(t)$ is a costate variable also belonging to the Hilbert space V and D^* and A^* denote adjoint operators of D and A, respectively.

Proof: By combining Eqs. (15) and (19), an augmented cost functional can be defined as

$$J_a(q) = \frac{1}{2T}\int_0^T [y(t) - z(t)]^T R(t)[y(t) - z(t)]\, dt + \int_0^T p(t)^T\left[\frac{d^2}{dt^2}u(t) + D\frac{du}{dt} + Au(t) - Bf(t)\right] dt \tag{23}$$

Following the variational approach, the necessary condition for the minimization of the equivalent cost function J_a is that its first variation vanishes for an arbitrary admissible parameter variation. The variation is

$$\delta J_a = \int_0^T p^T \frac{\partial}{\partial q}\left(D\frac{du}{dt} + Au - Bf\right)\delta q\, dt + \int_0^T \left[p^T\delta\left(\frac{d^2u}{dt^2}\right) + p^T D\delta\left(\frac{du}{dt}\right) + p^T A\delta u + \frac{1}{T}\delta u^T C^T R(Cu - z)\right] dt \tag{24}$$

The next step is the elimination of $\delta(d^2u/dt^2)$ and $\delta(du/dt)$ in Eq. (24).

By the repeated use of integration by parts,

$$\int_0^T p^T\delta\left(\frac{d^2u}{dt^2}\right) dt = p^T\delta\left(\frac{du}{dt}\right)\Big|_0^T - \int_0^T \frac{dp^T}{dt}\delta\left(\frac{du}{dt}\right) dt = p^T\delta\left(\frac{du}{dt}\right)\Big|_0^T - \frac{dp^T}{dt}\delta u\Big|_0^T + \int_0^T \frac{d^2p^T}{dt^2}\delta u\, dt \tag{25}$$

Since the initial conditions are given by Eq. (18), their variations are zero, i.e.,

$$\delta u(0) = \delta\left[\frac{d}{dt}u(0)\right] = 0 \tag{26}$$

By choosing the final conditions,

$$p(T) = \frac{d}{dt}p(T) = 0 \tag{27}$$

Eq. (25) becomes

$$\int_0^T p^T\delta\left(\frac{d^2u}{dt^2}\right) dt = \int_0^T \frac{d^2p^T}{dt^2}\delta u\, dt = \int_0^T \delta u^T \frac{d^2p}{dt^2} dt \tag{28}$$

Similarly,

$$\int_0^T p^T D\delta\left(\frac{du}{dt}\right) dt = p^T D\delta u\Big|_0^T - \int_0^T \frac{dp^T}{dt} D\delta u\, dt = -\int_0^T \frac{dp^T}{dt} D\delta u\, dt = -\int_0^T \delta u^T D^* \frac{dp}{dt} dt \tag{29}$$

where D^* is the adjoint operator of D defined by

$$\langle u, Dv\rangle = \langle D^*u, v\rangle \tag{30}$$

Substituting Eqs. (28) and (29) into Eq. (24), the variation becomes

$$\delta J_a = \int_0^T p^T \frac{\partial}{\partial q}\left(D\frac{du}{dt} + Au - Bf\right)\delta q\, dt + \int_0^T \delta u^T\left[\frac{d^2p}{dt^2} - D^*\frac{dp}{dt} + A^*p + \frac{1}{T}C^T R(Cu - z)\right] dt \tag{31}$$

where A^* is also the adjoint operator of A. In order to obtain a direct relationship between δJ_a and δq for an arbitrary δu, it is necessary to require the second integral to be zero. This results in the adjoint or costate differential equation

$$\frac{d^2}{dt^2}p(t) - D^*\frac{d}{dt}p(t) + A^*p(t) = -\frac{1}{T}C^T R(Cu - z) \tag{32}$$

with the final conditions

$$p(T) = \frac{d}{dt}p(T) = 0 \tag{33}$$

Finally, the variation in Eq. (31) becomes

$$\delta J_a = \int_0^T p^T \frac{\partial}{\partial q}\left(D\frac{du}{dt} + Au - Bf\right)\delta q\, dt \tag{34}$$

which is a direct relationship between the parameter variation δq and the cost variation.

Thus, the optimal parameter vector q^* satisfies both the system of Eqs. (15) and (16) and the costate equations (20) and (21); moreover, the first variation in Eq. (22) is zero.

Since the optimal parameter vector $q^* \in Q$ requires that the first variation δJ_a be zero, an iteration algorithm can be used to choose the parameter variation δq in the direction of decreasing δJ_a or in the negative gradient.

A Steepest Descent Computational Algorithm

A form of steepest descent or gradient projection method may be used to iteratively compute the optimal parameter vector q^*. The new value of q^{k+1} is computed as

$$q^{k+1}(x) = q^k - W^k\left(\frac{\delta J_a}{\delta q}\right) \tag{35}$$

where W^k is an arbitrary constant matrix, sufficiently small for ensuring convergence of the algorithm. There are a number of ways by which the matrix W^k can be chosen. It may be determined by a linear search so that $q^{k+1}(x)$ minimizes $J(q)$ in the direction $-\delta J_a/\delta q$ from $q^k(x)$. Alternatively and more simply, W^k can remain fixed for all k.

A flowchart for the iterative algorithm to estimate structural parameters is given in Fig. 1.

Parameter Identification of Vibrating Beams

This section gives some numerical results on the identification of parameters of a vibrating beam using the algorithm developed. Three specific cases of vibrating beams are considered. For numerical implementation of the algorithm, the finite-element method is employed to obtain approximate solutions of the state and costate equations of motion of the beam. In cases 1 and 2, the parameter estimations of a simply supported beam and a cantilevered beam are considered. In both the cases, the parameters are considered to be constants. In case 3, a beam parameter is considered to be a function of space variable. In all three cases, beams are excited by a step load function.

Case I: A Simply Supported Beam

In this section, the parameter estimation of a simply supported beam as shown in Fig. 2a is considered. The partial

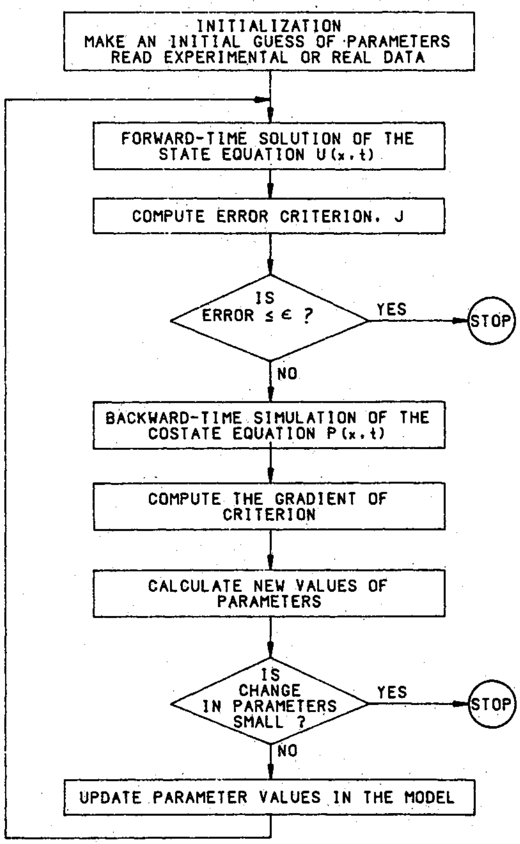

Fig. 1 Flow chart for the parameter estimation of a flexible structure.

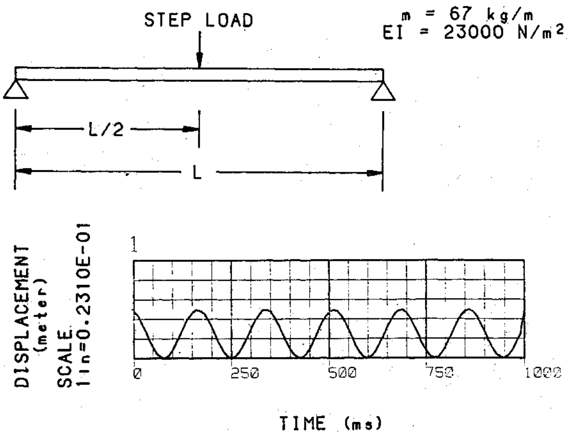

Fig. 2 Case I beam. a) Simply supported beam with a step load. b) Resultant displacements at $L/2$.

differential equation governing the displacement $u(x,t)$ of any section x at time t from its equilibrium is given by

$$\rho A \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = b(x)f(t), \qquad x \in [0,L],\ t > 0 \tag{36}$$

where ρ is the density of the material of the beam, A its cross-sectional area, E Young's modulus, and I the relevant second moment of the area of the cross section. The applied force is given by $f(t)$ and $b(x)$ is a function in x and represents the distributed nature of the force. Let $m = \rho A$ be the mass per unit length. Both the flexural rigidity EI and the mass per unit length m are considered constant. The length of the beam is L. The excitation force $f(t)$ is applied at a point, say, at the middle of the beam $L/2$. The boundary and the initial conditions for the simply supported beam are given as

$$u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) = 0, \qquad x \in \partial[0,L],\ t > 0 \tag{37}$$

and

$$u(x,0) = \frac{\partial}{\partial t} u(x,t)\Big|_{t=0} = 0, \qquad x \in [0,L] \tag{38}$$

It is assumed that a sensor is placed to directly measure displacements at some point on the beam, say, at $L/2$, i.e.,

$$y(t) = u\left(\frac{L}{2}, t\right) \tag{39}$$

Let the recorded data be denoted by $z(t)$.

$$z(t) = y(t) + e(t) \tag{40}$$

Then, the error criterion becomes

$$J = \frac{1}{2T} \int_0^T [y - z]^T R [y - z]\, \mathrm{d}t \tag{41}$$

Equation (36) can be rewritten as

$$\frac{\partial^2 u}{\partial t^2} = -\frac{EI}{m} \frac{\partial^4 u}{\partial x^4} + \frac{1}{m} \delta\left(x - \frac{L}{2}\right) f(t)$$

or

$$\frac{\partial^2 u}{\partial t^2} = -q_1 \frac{\partial^4 u}{\partial x^4} + q_2 \delta\left(x - \frac{L}{2}\right) f(t) \tag{42}$$

where $q_1 = EI/m$, $q_2 = 1/m$, and the parameter vector is defined by $q = [q_1, q_2]^T$.

From Eq. (32), the costate equation can be written as

$$\frac{\partial^2 p}{\partial t^2} = -q_1 \frac{\partial^4 p}{\partial x^4} + \frac{R}{T}[u - z]\, \delta\left(x - \frac{L}{2}\right), \qquad x \in [0,L],\ t \in [0,T) \tag{43}$$

From Eq. (33), the final conditions for the costate equation are

$$p(x,T) = \frac{\partial}{\partial t} p(x,t)\Big|_{t=T} = 0, \qquad x \in [0,L] \tag{44}$$

For a change in q_1 and q_2, it follows from Eq. (34) that

$$\delta J_a = \int_0^T \int_0^L p \frac{\partial^4 u}{\partial x^4} \delta q_1\, \mathrm{d}x\, \mathrm{d}t - \int_0^T \int_0^L p \delta\left(x - \frac{L}{2}\right) f(t) \delta q_2\, \mathrm{d}x\, \mathrm{d}t \tag{45}$$

The spatial integral in the first term of Eq. (45) can be written as

$$\int_0^L p \frac{\partial^4 u}{\partial x^4} \mathrm{d}x = p \frac{\partial^3 u}{\partial x^3}\Big|_0^L - \frac{\partial p}{\partial x} \frac{\partial^2 u}{\partial x^2}\Big|_0^L + \int_0^L \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 u}{\partial x^2} \mathrm{d}x \tag{46}$$

and by using the boundary conditions [Eq. (37)],

$$\int_0^T p \frac{\partial^4 u}{\partial x^4} \mathrm{d}x = \int_0^L \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 u}{\partial x^2} \mathrm{d}x \tag{47}$$

where the boundary conditions for the costate equation are chosen to be the same as those for the state equation, i.e.,

$$p(x,t) = \frac{\partial^2}{\partial x^2} p(x,t) = 0, \qquad x \in \partial[0,L],\ t \in [0,T] \tag{48}$$

Thus, the variation of Eq. (45) becomes

$$\delta J_a = \delta J_{a1} + \delta J_{a2} = \int_0^T \int_0^L \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 u}{\partial x^2} \delta q_1 \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_0^L p\delta\left(x - \frac{L}{2}\right) f(t) \delta q_2 \, \mathrm{d}x \, \mathrm{d}t \tag{49}$$

To apply the steepest descent algorithm [Eq. (35)], the gradient vector for a uniform beam is computed as

$$\frac{\delta J_a}{\delta q_1} = \int_0^T \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 u}{\partial x^2} \mathrm{d}t$$

and

$$\frac{\delta J_a}{\delta q_2} = -\int_0^T p\delta\left(x - \frac{L}{2}\right) f(t) \, \mathrm{d}t \tag{50}$$

Thus, parameters can be updated by the steepest descent algorithm

$$q_i^{k+1} = q_i^k - w_i \left(\frac{\delta J_a}{\delta q_i}\right)^k, \qquad i = 1,2 \tag{51}$$

where w_1 and w_2 are arbitrary weighting functions chosen sufficiently small to ensure the convergence of the algorithm.

The simply supported beam and its load are shown in Fig. 2a. The resulting displacement is shown in Fig. 2b. The beam is 4 m long and a step load of 100 N/m is applied at the midpoint. The true values of mass per unit length m and flexural rigidity EI are 67 kg/m and 2300 N/m^2, respectively. A point sensor is also located at the midpoint to measure the displacement. The weighting factors used for convergence are $w_1 = 150{,}000$ and $w_2 = 0.03$. The performance data are presented in Table 2.

Case II: A Cantilevered Beam

To further illustrate the method, a cantilevered beam of Fig. 3a is considered. This time, a damping term is added to Eq. (36),

$$m\frac{\partial^2 u}{\partial t^2} - 2\xi\sqrt{mEI}\frac{\partial^3 u}{\partial x^2 \partial t} + EI\frac{\partial^4 u}{\partial x^4} = b(x)f(t)$$

$$x \in [0,L], \; t > 0 \tag{52}$$

where ξ is a damping coefficient.

A step load is applied at the tip of the beam, i.e., at $x = L$ and the displacement data are also gathered at L, i.e.,

$$y(t) = u(L,t)$$

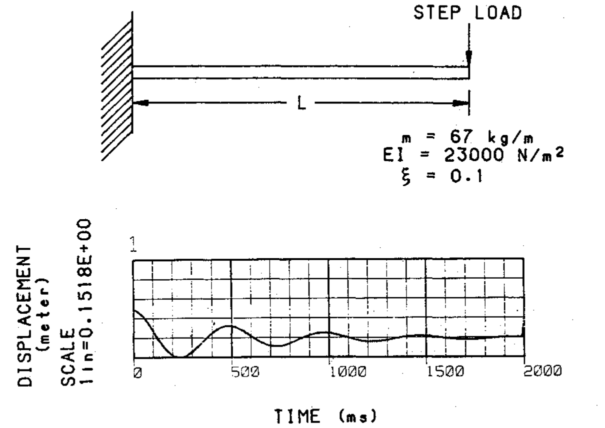

Fig. 3 Case II beam. a) Cantilevered beam with a step load. b) Resultant displacements at *L*.

The boundary conditions for a cantilevered beam are

$$u(0,t) = \frac{\partial^2}{\partial x^2} u(x,t)\bigg|_{x=0} = 0, \qquad t > 0$$

$$\frac{\partial^2}{\partial x^2} u(x,t)\bigg|_{x=L} = \frac{\partial^3}{\partial x^3} u(x,t)\bigg|_{x=L} = 0, \qquad t > 0 \tag{53}$$

The beam is initially at rest and hence the initial conditions are

$$u(x,0) = \frac{\partial}{\partial t} u(x,t)\bigg|_{t=0} = 0, \qquad x \in [0,L]$$

$$\frac{\partial u(x,t)}{\partial x}\bigg|_{t=0} = \frac{\partial^2}{\partial x^2} u(x,t)\bigg|_{t=0} = 0, \qquad x \in [0,L] \tag{54}$$

The error criterion is

$$J = \frac{1}{2T}\int_0^T [y - z]^T R[y - z] \, \mathrm{d}t$$

Equation (52) can be rewritten as

$$\frac{\partial^2 u}{\partial t^2} = q_3 \frac{\partial^3 u}{\partial x^2 \partial t} - q_1 \frac{\partial^4 u}{\partial x^4} + q_2 \delta(x - L) f(t) \tag{55}$$

where

$$q_1 = \frac{EI}{m}, \quad q_2 = \frac{1}{m}, \quad q_3 = 2\xi\sqrt{\frac{EI}{m}}$$

and the parameter vector is defined by $q = [q_1, q_2, q_3]^T$.

Table 2 Performance data for case I

Iteration	q_1	q_2	m	EI	$\int_0^T \mathrm{error}^2 \, \mathrm{d}t$
1	301.58	0.0158	63.00	19000.00	0.20178E-01
2	311.68	0.0154	64.84	20207.72	0.16640E-01
3	325.65	0.0151	66.27	21582.87	0.71528E-02
4	338.48	0.0149	66.91	22647.68	0.57208E-03
5	342.11	0.0149	66.98	22913.64	0.34507E-04
6	342.96	0.0149	66.99	22973.13	0.26524E-04
7	343.19	0.0149	66.99	22989.35	0.24364E-04
True values	343.28	0.0149	67.00	23000.00	

From Eq. (32), the costate equation can be written as

$$\frac{\partial^2 p}{\partial t^2} = q_3 \frac{\partial^3 p}{\partial x^2 \partial t} - q_1 \frac{\partial^4 p}{\partial x^4} - \frac{R}{T}[y - z]\delta(x - L)$$

$$x \in [0,L],\ t \in [0,T) \quad (56)$$

From Eq. (33), the final conditions for the costate equation are

$$p(x,T) = \frac{\partial}{\partial t} p(x,t)\bigg|_{t=T} = 0, \qquad x \in [0,L] \quad (57)$$

For a change in q_1, q_2, and q_3, it follows from Eq. (34) that

$$\delta J_a = \int_0^T \int_0^L p \frac{\partial^4 u}{\partial x^4} \delta q_1 \, dx\, dt - \int_0^T \int_0^L p\delta(x - L) f(t) \delta q_2 \, dx\, dt - \int_0^T \int_0^L p \frac{\partial^3 u}{\partial^2 x \partial t} \delta q_3 \, dx\, dt$$

Following the similar procedure used in obtaining Eq. (49), the variation becomes

$$\delta J_a = \delta J_{a1} + \delta J_{a2} + \delta J_{a3}$$
$$= \int_0^T \int_0^L \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 u}{\partial x^2} \delta q_1 \, dx\, dt - \int_0^T \int_0^L p\delta(x - L) f(t) \delta q_2 \, dx\, dt - \int_0^T \int_0^L p \frac{\partial^3 u}{\partial^2 x \partial t} \delta q_3 \, dx\, dt \quad (58)$$

where the boundary conditions for the costate equation are the same as those for the state equation, i.e.,

$$p(0,t) = \frac{\partial^2}{\partial x^2} p(x,t)\bigg|_{x=0} = 0, \qquad t > 0$$
$$\frac{\partial^2}{\partial x^2} p(x,t)\bigg|_{x=L} = \frac{\partial^3}{\partial x^3} p(x,t)\bigg|_{x=L} = 0, \qquad t > 0 \quad (59)$$

The equation for obtaining gradients for q_1 and q_2 are the same ones for case I. The gradient for q_3 can be obtained by simplifying the additional gradient of cost term due to damping,

$$\delta J_{a3} = -\int_0^T \int_0^L p \frac{\partial^3 u}{\partial^2 x \partial t} \delta q_3 \, dx\, dt$$

Since

$$-\int_0^T p \frac{\partial^3 u}{\partial x^2 \partial t} dt = -p \frac{\partial^2 u}{\partial x^2}\bigg|_0^T + \int_0^T \frac{\partial p}{\partial t} \frac{\partial^2 u}{\partial x^2} dt$$

by using the initial and final conditions [Eqs. (54) and (57)], respectively,

$$\delta J_{a3} = \int_0^T \int_0^L \frac{\partial p}{\partial t} \frac{\partial^2 u}{\partial x^2} \delta q_3 \, dx\, dt \quad (60)$$

To apply the steepest descent algorithm [Eq. (35)], the gradient vector for a uniform beam is computed as

$$\frac{\delta J_a}{\delta q_3} = \int_0^T \frac{\partial p}{\partial t} \frac{\partial^2 u}{\partial x^2} dt \quad (61)$$

Thus, parameters can be updated as before by the steepest descent algorithm [Eq. (35)] for $i = 1,2,3$.

For the second case the cantilevered beam shown in Fig. 3a is excited at the tip by a step load function and the resultant displacement is shown in Fig. 3b. Here, the true values of parameters are $m = 67$ kg/m, $EI = 2300$ N/m^2, and $\xi = 0.1$. The weighting factors used for the convergence are $w_1 = 3500$, $w_2 = 0.008$, and $w_3 = 125$. The performance data are presented in Table 3.

Case III: A Simply Supported Beam with Spatially Variable Parameter

So far, we have demonstrated the effectiveness of our algorithm to the cases where beam parameters do not vary with the space variable. In this example, the identification algorithm is applied to the case of a simply supported beam with spatially variable flexural rigidity as shown in Fig. 4a. For simplicity, it is assumed that mass per unit length is constant and the damping coefficient is zero. The equation of motion of a normalized beam is given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[q(x)\frac{\partial^2 u}{\partial x^2}\right] = b(x) f(t), \qquad x \in [0, L],\ t > 0 \quad (62)$$

where $q = EI(x)$.

The boundary and the initial conditions are as given in Eqs. (37) and (38).

From Eq. (32), the costate equation can be written as

$$\frac{\partial^2 p}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left[q(x)\frac{\partial^2 p}{\partial x^2}\right] - \frac{R}{T}[u - z]$$

$$x \in [0,L],\ t \in [0, T) \quad (63)$$

The boundary and final conditions for the costate equation are the same as those given in Eqs. (44) and (48).

Table 3 Performance data for case II

Iteration	q_1	q_2	q_3	m	EI	ξ	$\int_0^T \text{error}^2\, dt$
1	301.59	0.0159	2.08	63.00	19000.00	0.060	0.7843E-01
2	318.01	0.0155	2.76	64.50	20512.92	0.077	0.1664E-01
3	326.20	0.0153	3.06	65.26	21286.34	0.085	0.9432E-02
4	331.30	0.0152	3.23	65.73	21775.47	0.089	0.4498E-02
5	334.72	0.0151	3.34	66.05	22107.25	0.091	0.2284E-02
6	337.10	0.0151	3.43	66.27	22339.85	0.093	0.1238E-02
7	338.83	0.0151	3.49	66.43	22510.09	0.095	0.6649E-03
8	340.07	0.0150	3.53	66.55	22632.61	0.096	0.3796E-03
9	340.96	0.0150	3.55	66.65	22721.78	0.096	0.2199E-03
10	341.56	0.0150	3.56	66.71	22783.93	0.096	0.1519E-03
11	342.06	0.0150	3.59	66.76	22835.96	0.097	0.8667E-04
12	342.44	0.01497	3.62	66.80	22874.19	0.098	0.4961E-04
13	342.73	0.01496	3.64	66.83	22903.95	0.098	0.1165E-04
True values	343.28	0.0149	3.70	67.00	23000.00	0.1	

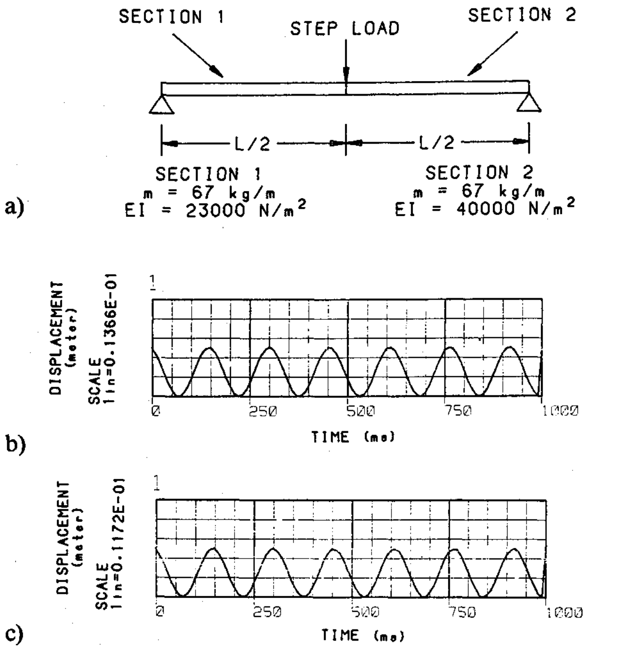

Fig. 4 Case III beam. a) Simply supported beam with spatially variable flexible rigidity. b) Resultant displacements at $L/4$. c) Resultant displacements at $3L/4$.

Table 4 Performance data for case III

Iteration	q (section 1)	q (section 2)	$\int_0^T \text{error}^2\, \mathrm{d}t$
1	21000.00	39000.00	0.76863E-02
2	21686.15	39286.47	0.40601E-02
3	22261.30	39521.45	0.14636E-02
4	22630.66	39688.41	0.41442E-03
5	22814.49	39759.51	0.13629E-03
6	22911.04	39793.95	0.51151E-04
7	22968.10	39815.07	0.11209E-04
True values	23000.00	40000.00	

For a change in q, it follows from Eq. (34) that

$$\delta J_a = \int_0^T \int_0^L p \frac{\partial}{\partial q} \left\{ \frac{\partial^2}{\partial x^2} \left[q(x) \frac{\partial^2 u}{\partial x^2} \right] \right\} \delta q \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T \int_0^L p \frac{\partial}{\partial} \left(\frac{\partial^2 q}{\partial x^2} \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial q}{\partial x} \frac{\partial^3 u}{\partial x^3} + q \frac{\partial^4 u}{\partial x^4} \right) \delta q \,\mathrm{d}x\,\mathrm{d}t \tag{64}$$

From which one gets

$$\delta J_a = \int_0^T \int_0^L p \frac{\partial^4 u}{\partial x^4} \delta q \,\mathrm{d}x\,\mathrm{d}t$$

which may be further simplified as

$$\delta J_a = \int_0^T \int_0^L \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 u}{\partial x^2} \delta q \,\mathrm{d}x\,\mathrm{d}t \tag{65}$$

Hence, the gradient vector for a section x of the beam is

$$\frac{\delta J_a(x)}{\delta q} = \int_0^T \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 u}{\partial x^2} \mathrm{d}t \tag{66}$$

Thus, parameters can be updated by the steepest descent algorithm as in the previous cases.

The simply supported beam and its load, shown in Fig. 4a is divided into two sections. The flexural rigidity coefficients q for sections 1 and 2 are, respectively, 2300 and 4000 N/m². The mass per unit length m is 67 kg/m and assumed constant throughout the length of the beam. The beam is excited at $L/2$ and the displacements are measured by two point sensors located at $L/4$ and $3L/4$. The weighting factors used for convergence are $w_1 = 125$ for section 1 and $w_2 = 100$ for section 2. The performance data are presented in Table 4.

Conclusion

Infinite-dimensional identification method presented in this paper shows a significant promise in the parameter estimation of flexible beams with potential for applications to large space structures. The basic approach is the abstract formulation of the system dynamics in function spaces and then applying optimal control theory to adjust the system parameters so that the error between actual and model data is minimized. The use of partial differential equations for the purpose of estimation may eliminate many problems associated with model truncation in the finite-dimensional approach. Based on partial differential equation models and a quadratic performance index, an algorithm to estimate the optimal parameters of the equivalent model has been developed. The numerical results show the effectiveness of the algorithm in estimating beam parameters in three specific cases. The results show a fairly good match between the model and the estimated parameters.

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